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# Topological susceptibility and string tension in $\text{CP}^{N-1}$ models \*

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## Abstract

We determined the topological susceptibility and the string tension in the lattice  $\text{CP}^{N-1}$  models for a wide range of values of  $N$ , in particular for  $N = 4, 10, 21, 41$ . Quantitative agreement with the large- $N$  predictions is found for the  $\text{CP}^{20}$  and the  $\text{CP}^{40}$  models.

## 1 INTRODUCTION

The most attractive feature of two dimensional  $\text{CP}^{N-1}$  models is their similarity with the Yang-Mills theories in four dimensions. Most properties of  $\text{CP}^{N-1}$  models have been obtained in the context of the  $1/N$  expansion around the large- $N$  saddle point solution. An alternative and more general non-perturbative approach is the simulation of the theory on the lattice. It is then important to perform Monte Carlo simulations of the lattice  $\text{CP}^{N-1}$  models to check the validity and the range of applicability of the  $1/N$  expansion.

The large- $N$  expansion predicts an exponential area law behavior for sufficiently large Wilson loops, which implies confinement, with a string tension  $O(1/N)$ , and absence of screening due to the dynamical matter fields.  $\text{CP}^{N-1}$  models have a non-trivial topological structure; at large  $N$  the topological susceptibility turns out to be  $O(1/N)$ . In this talk the features of  $\text{CP}^{N-1}$  models concerning confinement and topology are investigated. In order to study the approach to the large- $N$  asymptotic regime, we performed numerical simulations for a wide range of values of  $N$ , in particular  $N = 4, 10, 21, 41$ . A more detailed discussion of these simulations and of their results can be found in Refs. [1, 2, 3].

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## 2 LATTICE FORMULATION

We regularize the theory on the lattice by considering the following action:

$$S_g = -N\beta \sum_{n,\mu} (\bar{z}_{n+\mu} z_n \lambda_{n,\mu} + \bar{z}_n z_{n+\mu} \bar{\lambda}_{n,\mu}), \quad (1)$$

where  $z_n$  is an  $N$ -component complex scalar field, satisfying  $\bar{z}_n z_n = 1$ , and  $\lambda_{n,\mu}$  is a  $U(1)$  gauge field. We also considered its tree Symanzik improved counterpart  $S_g^{\text{Sym}}$  in order to test universality.

The standard correlation length  $\xi_w$  is extracted from the long-distance behavior of the zero space momentum correlation function  $G_P$  of two projector operators  $P_{ij}(x) = \bar{z}_i(x) z_j(x)$ . We define an alternative correlation length  $\xi_G$  from the second moment of the correlation function  $G_P$ . For  $N = 2$   $\xi_G/\xi_w \simeq 1$  within 1% [4], while the large- $N$  expansion predicts [5]

$$\frac{\xi_G}{\xi_w} = \sqrt{\frac{2}{3}} + O\left(\frac{1}{N^{2/3}}\right). \quad (2)$$

Unlike  $\xi_w$ , which is a non-analytic function of  $1/N$  around  $N = \infty$ ,  $\xi_G$  can be expand in powers of  $1/N$  [5].

## 3 TOPOLOGY

The large- $N$  predictions concerning the topological susceptibility  $\chi_t$  are [6]

$$\chi_t \xi_G^2 = \frac{1}{2\pi N} \left(1 - \frac{0.3801}{N}\right) + O\left(\frac{1}{N^3}\right), \quad (3)$$

and [7]

$$\chi_t \xi_w^2 = \frac{3}{4\pi N} + O\left(\frac{1}{N^{5/3}}\right). \quad (4)$$

Eq. (3) and Eq. (4) are not in contradiction with each other due to Eq. (2), but the first one should be testable at lower values of  $N$  according to the powers of  $N$  in the neglected terms.

A troublesome point in the lattice simulation technique is the study of the topological properties and the determination of  $\chi_t$ . While for large  $N$  the geometrical definition  $\chi_t^g$  [8] is expected to reproduce the physical topological susceptibility, at low  $N$   $\chi_t^g$  could receive unphysical contributions from exceptional configurations, called dislocations. The dislocation contributions may either survive in the continuum limit, as it happens for the  $CP^1$  or  $O(3)$   $\sigma$  model, or push the scaling region for  $\chi_t^g$  to very large  $\beta$  values.

Another approach, the field theoretical method, relies on a definition of topological charge density by a local polynomial in the lattice variables  $q_L$ . The

correlation at zero momentum of two  $q^L$  operators  $\chi_t^L$  is related to  $\chi_t$  through the following equation:

$$\begin{aligned}\chi_t^L(\beta) &= a^2 Z(\beta)^2 \chi_t + a^2 A(\beta) \langle S(x) \rangle \\ &+ P(\beta) \langle I \rangle + O(a^4).\end{aligned}\tag{5}$$

$S(x)$  is the trace of the energy-momentum tensor,  $I$  is the identity operator.  $Z(\beta)$ ,  $P(\beta)$ , and  $A(\beta)$  are ultraviolet effects, since they originate from the ultraviolet cutoff-dependent modes. The field theoretical method consists in measuring  $\chi_t^L(\beta)$ , evaluating  $Z(\beta)$ ,  $A(\beta)$  and  $P(\beta)$ , and using Eq. (5) to extract  $\chi_t$ .

A third method, called cooling method, measures  $\chi_t$  on an ensemble of configurations cooled by locally minimizing the action [12].

### 3.1 The $CP^3$ model

We determined  $\chi_t$  by performing simulations with both actions  $S_g$  and  $S_g^{\text{Sym}}$  and by using different methods. In Fig. 1 the dimensionless quantity  $\chi_t \xi_G^2$  is plotted versus  $\xi_G$ . Data obtained by the geometrical method, indicated by the filled symbols in Fig. 1, violate universality, showing that  $N = 4$  is not large enough to suppress the unphysical configurations contributing to  $\chi_t^g$ , at least for  $\xi \leq 30$ .

To apply the field theoretical method, we considered the following lattice operator:

$$q^L(x) = -\frac{i}{2\pi} \sum_{\mu\nu} \epsilon_{\mu\nu} \text{Tr} [P(x) \Delta_\mu P(x) \Delta_\nu P(x)],\tag{6}$$

where  $\Delta_\mu$  is a symmetrized version of the finite derivative. We neglected the contribution of the mixing with  $S(x)$ ; this assumption is supported by perturbative arguments. We obtained non-perturbative estimates of  $Z(\beta)$  and  $P(\beta)$  by using the “heating method” described in Ref. [11]. This method relies on the distinction between the fluctuations at  $l \sim a$ , contributing to the renormalizations, and those at  $l \sim \xi$  determining the relevant topological properties. Due to the critical slow down phenomenon, fluctuations at  $l \sim a$  are soon thermalized, while, for large  $\xi$ , the topological charge thermalization is much slower, allowing a direct determination of  $Z(\beta)$  (when heating an instanton configuration) and  $P(\beta)$  (when heating the flat configuration).

An independent measure of  $\chi_t$  is obtained using the cooling method. The results of both field theoretical and cooling methods are shown in Fig. 1. Scaling, universality, and good agreement between the two methods are observed.

In conclusion, we quote for the  $CP^3$  model  $\chi_t \xi_G^2 \simeq 0.06$  with an uncertainty of about 10%.

### 3.2 The $\text{CP}^9$ model

We measured  $\chi_t$  by using the geometrical definition. Data were taken for  $\xi_G$  up to about 10 lattice units. Data for  $S_g$  show a slow approach to scaling, while for  $S_g^{\text{Sym}}$  a better behavior is observed. For  $S_g$  the leading scaling violation term must be  $O(\ln \xi/\xi^2)$  when  $\xi \rightarrow \infty$  [9]. For the tree Symanzik improved actions the leading logarithmic corrections are absent, and scale violations are  $O(\xi^{-2})$  [10]. In Fig. 2 we plot  $\chi_t^g \xi_G^2$  versus  $\ln \xi_G/\xi_G^2$ . Assuming that the scaling violation term proportional to  $\ln \xi/\xi^2$  is already dominant in our range of correlation lengths, we extrapolated data of  $\chi_t^g$  for the action  $S_g$ . We found  $\chi_t^g \xi_G^2 = 0.0174(12)$ , which is in agreement with the value of  $\chi_t^g \xi_G^2$  obtained with the action  $S_g^{\text{Sym}}$ . We then conclude that for the  $\text{CP}^9$  model  $\chi_t^g$  is a good estimator of the topological susceptibility.

### 3.3 The $\text{CP}^{20}$ and $\text{CP}^{40}$ models

As shown in Fig. 3 for both the  $\text{CP}^{20}$  and  $\text{CP}^{40}$  models, data of  $\chi_t$ , obtained by using the geometrical method, are consistent with the large- $N$  prediction (3), whose results are indicated by the dashed lines. Fitting the data for the  $\text{CP}^{20}$  model, we found  $\chi_t \xi_G^2 = 0.0076(3)$ , to be compared with the value  $\chi_t \xi_G^2 = 0.00744$  coming from Eq. (3). However, the above result still disagrees with Eq. (4), which would require  $\xi_G^2/\xi_w^2 \simeq 2/3$ , while we found  $\xi_G^2/\xi_w^2 \simeq 0.91$  in our simulations.

In Fig. 4 we summarize our results by plotting  $\chi_t \xi_G^2$  versus  $1/N$ , showing how the topological susceptibility approaches the large- $N$  asymptotic behavior (3), represented by the solid line.

## 4 CONFINEMENT

The large- $N$  prediction for the string tension  $\sigma$  is

$$\sigma \xi_G^2 = \frac{\pi}{N} + O\left(\frac{1}{N^2}\right). \quad (7)$$

The string tension can be easily extracted by measuring the Creutz ratios defined by

$$\chi(R, T) = \ln \frac{W(R, T-1) W(R-1, T)}{W(R, T) W(R-1, T-1)}. \quad (8)$$

where  $W(R, T)$  are the Wilson loops constructed with the  $\lambda_\mu$  field. In a 2-d finite lattice with periodic boundary conditions, the large abelian Wilson loops of a confining theory are subject to large finite size effects. For sufficiently large  $R$  the behavior of the Creutz ratios  $\eta(R) \equiv \chi(R, R)$ , i.e. of those with equal

arguments, should be [1]:

$$\eta(R) \simeq \sigma \left[ 1 - \left( \frac{2R-1}{L} \right)^2 \right], \quad (9)$$

where  $L$  is the lattice size. To compare data from different lattices it is convenient to define a rescaled Creutz ratio

$$\eta_r(R) = \eta(R) \left[ 1 - \left( \frac{2R-1}{L} \right)^2 \right]^{-1} \simeq \sigma. \quad (10)$$

In Fig. 5 we plot  $\eta_r(R)\xi_G^2$  versus the physical distance  $r = R/\xi_G$  for the  $\text{CP}^9$ ,  $\text{CP}^{20}$ , and  $\text{CP}^{40}$  models. Starting from  $r \simeq 2$  the rescaled Creutz ratios show a clear plateau which is the evidence of the string tension. A similar behavior is observed for  $N = 4$ . We do not see evidence of screening effects (at least up to  $r \simeq 3\xi_G$ , data for larger  $r$  were too noisy) confirming the picture coming from the  $1/N$  expansion. Data for the  $\text{CP}^{20}$  and  $\text{CP}^{40}$  show a good agreement with the large- $N$  prediction (7), represented by the dashed lines in Fig. 5.

## 5 CONCLUSION

With regard to topology and confinement, numerical results show a qualitative agreement with the continuum  $1/N$  expansion for all values of  $N$ , while quantitative agreement is found for  $N = 21$  and for higher values of  $N$ . On the other hand, the approach to the large- $N$  asymptotic regime of quantities involving the mass gap appears very slow and the  $\text{CP}^{40}$  model should be still outside the region where the complete mass spectrum predicted by the  $1/N$  expansion could be observed [2]. The agreement in this sector of the theory is expected to be reached at very large  $N$ , because of the large coefficient in the effective expansion parameter  $6\pi/N$  that can be extracted from a non-relativistic Schrödinger equation analysis of the linear confining potential [5].

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## Figure captions

**Fig. 1:**  $\chi_t \xi_G^2$  versus  $\xi_G$  for the  $\text{CP}^3$  model. We plot data of the geometrical method (for  $S_g$  “ $\square$ ” and  $S_g^{\text{Sym}}$  “ $\diamond$ ”), the field theoretical method ( $S_g$  “ $\diamond$ ”,  $S_g^{\text{Sym}}$  “ $\times$ ”), and the cooling method ( $S_g$  “ $\square$ ”,  $S_g^{\text{Sym}}$  “ $\circ$ ”).

**Fig. 2:**  $\chi_t \xi_G^2$  versus  $\ln \xi_G / \xi_G^2$  for the  $\text{CP}^9$  model.

**Fig. 3:**  $\chi_t \xi_G^2$  for the  $\text{CP}^{20}$  and the  $\text{CP}^{40}$  models.

**Fig. 4:**  $\chi_t \xi_G^2$  versus  $1/N$ .

**Fig. 5:**  $\eta_r(R) \xi_G^2$  versus  $r = R/\xi_G$ , for  $N = 10$ ,  $N = 21$ , and  $N = 41$ . For each  $N$  we show data at two values of  $\beta$ .